3 Stochastic Integrals

Exercise 3.1

(a) Since $Z(t)$ is determinist, we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha t} dt \\ &= \alpha Z(t) dt. \end{aligned}$$

(b) By definition of a stochastic differential

$$dZ(t) = g(t) dW(t)$$

(c) Using Itô's formula

$$\begin{aligned} dZ(t) &= \frac{\alpha^2}{2} e^{\alpha W(t)} dt + \alpha e^{\alpha W(t)} dW(t) \\ &= \frac{\alpha^2}{2} Z(t) dt + \alpha Z(t) dW \end{aligned}$$

(d) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha x} dX(t) + \frac{\alpha^2}{2} e^{\alpha x} (dX(t))^2 \\ &= Z(t) \left[\alpha\mu + \frac{1}{2}\alpha^2\sigma^2 \right] dt + \alpha\sigma Z(t) dW(t). \end{aligned}$$

(e) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= 2X(t)dX(t) + (d(X(t))^2 \\ &= Z(t) \left[2\alpha + \sigma^2 \right] dt + 2Z\sigma dW(t). \end{aligned}$$

Exercise 3.3 By definition we have that the dynamics of $X(t)$ are given by $dX(t) = \sigma(t) dW(t)$.

Consider $Z(t) = e^{iuX(t)}$. Then using the Itô's formula we have that the dynamic of $Z(t)$ can be described by

$$dZ(t) = \left[-\frac{u^2}{2}\sigma^2(t) \right] Z(t) dt + \left[iu\sigma(t) \right] Z(t) dW(t)$$

From $Z(0) = 1$ we get,

$$Z(t) = 1 - \frac{u^2}{2} \int_0^t \sigma^2(s) Z(s) ds + iu \int_0^t \sigma(s) Z(s) dW(s).$$

Taking expectations we have,

$$\begin{aligned} E[Z(t)] &= 1 - \frac{u^2}{2} E\left[\int_0^t \sigma^2(s) Z(s) ds\right] + iuE\left[\int_0^t \sigma(s) Z(s) dW(s)\right] \\ &= 1 - \frac{u^2}{2}\left[\int_0^t \sigma^2(s) E[Z(s)]\, ds\right] + 0 \end{aligned}$$

By setting $E[Z(t)] = m(t)$ and differentiating with respect to t we find an ordinary differential equation,

$$\frac{\partial m(t)}{\partial t} = -\frac{u^2}{2} m(t) \sigma^2(t)$$

with the initial condition $m(0) = 1$ and whose solution is

$$\begin{aligned} m(t) &= \exp\left\{-\frac{u^2}{2}\int_0^t \sigma 2(s) ds\right\} \\ &= E[Z(t)] \\ &= E\left[e^{iuX(t)}\right] \end{aligned}$$

So, $X(t)$ is normally distributed. By the properties of the normal distribution the following relation

$$E\left[e^{iuX(t)}\right] = e^{iuE[X(t)] - \frac{u^2}{2} V[X(t)]}$$

where $V[X(t)]$ is the variance of $X(t)$, so it must be that $E[X(t)] = 0$ and $V[X(t)] = \int_0^t \sigma^2(s) ds$.

Exercise 3.5 We have a sub martingale if $E[X(t)|\mathcal{F}_s] \geq X(s) \forall, t \geq s$. From the dynamics of X we can write

$$X(t) = X(s) + \int_s^t \mu(z) dz + \int_s^t \sigma(z) dW(z).$$

By taking expectation, conditioned at time s, from both sides we get

$$\begin{aligned} E[X(t)|\mathcal{F}_s] &= E[X(s)|\mathcal{F}_s] + E\left[\left.\int_s^t \mu(z) dz\right| \mathcal{F}_s\right] \\ &= X(s) + E^s\left[\left.\underbrace{\int_s^t \mu(z) dz}_{\geq 0}\right| \mathcal{F}_s\right] \\ &\geq X(s) \end{aligned}$$

so X is a sub martingale.

Exercise 3.6 Set $X(t) = h(W_1(t), \cdots, W_n(t))$.

We have by Itô that

$$dX(t) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)$$

where $\frac{\partial h}{\partial x_i}$ denotes the first derivative with respect to the i-th variable, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ denotes the second order cross-derivative between the i-th and j-th variable and all derivatives should be evaluated at $(W_1(s), \cdots, W_n(s))$.

Since we are dealing with independent Wiener processes we know

$$\forall u : \quad dW_i(u) dW_j(u) = 0 \text{ for } i \neq j \quad \text{and} \quad dW_i(u) dW_j(u) = du \text{ for } i = j,$$

so, integrating we get

$$\begin{aligned} X(t) &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(u) dW_j(u) \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \left[dW_i(u) \right]^2 \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du. \end{aligned}$$

Taking expectations

$$\begin{aligned} E\left[X(t) \middle| \mathcal{F}_s \right] &= E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right] + E\left[\frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\ &= \underbrace{\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^s \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du}_{X(s)} \\ &\quad + \underbrace{E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right]}_{0} + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\ &= X(s) + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right]. \end{aligned}$$

- If h is *harmonic* the last term is zero, since $\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} = 0$, we have

$$E\left[X(t) \middle| \mathcal{F}_s \right] = X(s) \quad \text{so } X \text{ is a martingale.}$$